Introduction to Mathematical Methods in Physics(*)

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(*) Not all of them

Introduction : the course in the workshop

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Introduction to Mathematical methods in Physics	Mathieu Aurousseau (UJ)
Introduction to Quantum Field Theory	Cesareo Dominguez (UCT)
Introduction to Experimental Particle Physics	Ketevi Assamagan (BNL & UJ)
Introduction to Feynman Diagrams	Robert de Mello Koch (Wits)
Introduction to Advanced Statistical Physics	Tamas Biro (MTKFA, Hungary)
Introduction to Particle Physics	Azwinndini Muronga (UJ)
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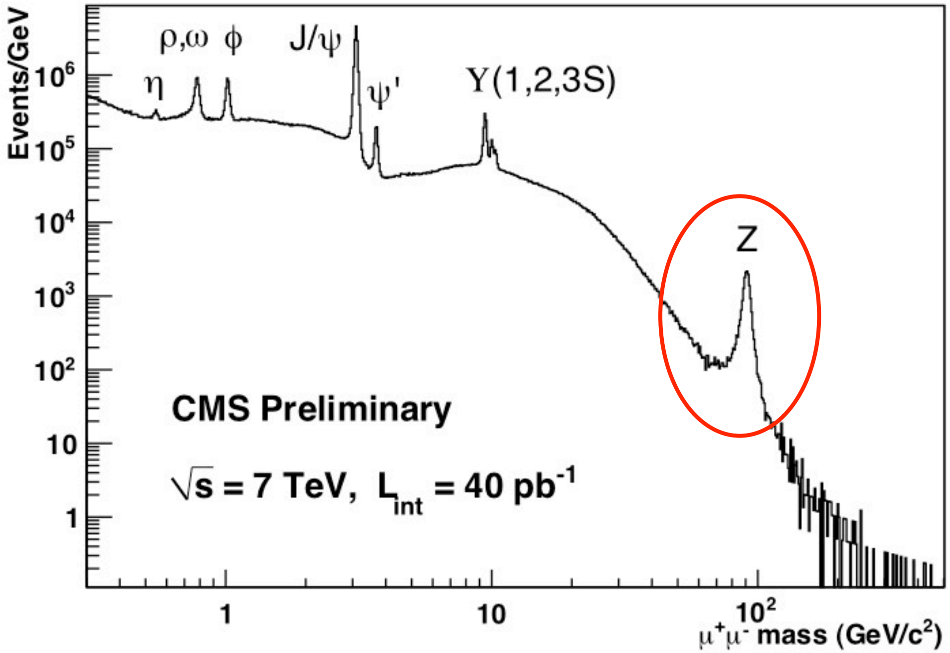

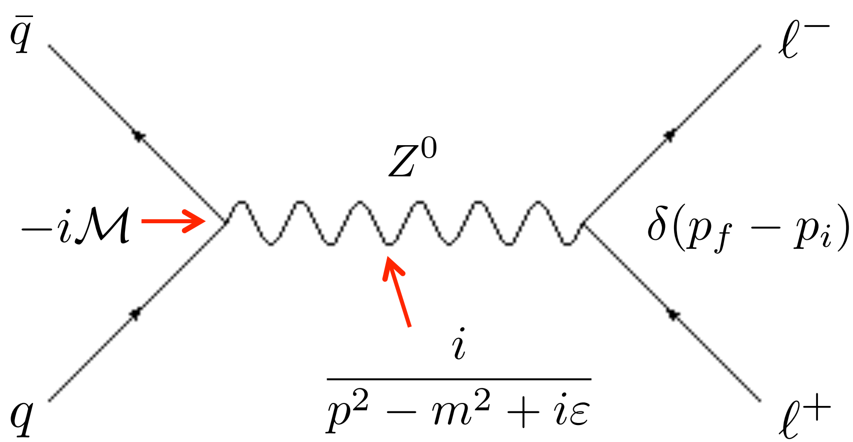

Use Mathematical "tools"

- Group theory, symmetries
- Complex numbers
- Matrices
- Dirac distribution
- Functions
- Statistics
- Probabilities
- Calculus, algebra

… in a Physics "context"

Introduction : disclaimers

- Why Mathematical “tools” ?
  - Physicists mostly use Maths as a toolbox
  - No “pure Maths” thoughts :
    - physicist : 1 solution to 1 problem
    - All solutions to a class of problems : this is the mathematician job – already done

- Why in a Physics “context” ?
  - (Almost) no demonstrations (not the point)
  - Sometimes, simple statements hide a lot of mathematical work
  - “Well-behaved” objects (numbers, functions)
  - Existing solutions
  - Approximations
  - The Physical world is real, not complex

- However :
  - Mathematics are very powerful *if done properly* :
    - be cautious, make drawings, write down things, justify approximations, quote theorems
    - sometimes use general results and narrow down to your particular problems
    - always check the consistency of your *physics* results
    - error signs can be embarrassing (at best) – or dramatic

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- *I- Introduction to Complex Analysis*
  - The complex algebra
  - Functions of a complex variable
  - Derivation and integration, Cauchy theorems
  - Residue theorems, Jordan lemma
  - One example : the Klein-Gordon propagator

- *II- Distributions, the Dirac δ "function"*

INTRODUCTION TO COMPLEX ANALYSIS

At the end of these lectures, you should …

- understand what are complex numbers and functions
- know how to determine whether a function is analytic or not, etc.
- be able to calculate a complex integral using the Residue theorem

Course mainly based on Arfken&Weber, *“Mathematical methods for physicists”*

Outline

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- Introduction : the complex algebra
  - Complex numbers
  - Operation on complex numbers
- Function of a complex variable
  - Generalizing real functions
  - Differentiation of complex functions
  - Analytic and entire functions
  - Singularities, poles, cuts
- Integration, important theorems
  - Integration in the complex plane
  - Cauchy's theorems
  - Residues theorem, Jordan lemma
- Practical example
  - The Klein-Gordon propagator

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$$\boxed{i^2 = -1}$$

- $\mathbb{C} \equiv \{(x, y) \mid x, y \in \mathbb{R}\}$ $\quad i \equiv (0, 1)$
- $\mathbb{C} \equiv \{x + iy \mid x, y \in \mathbb{R}\}$ $\quad i \equiv 0 + i \cdot 1$
- $\mathbb{C} \equiv \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$ $\quad i \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Addition and multiplication are *defined* as :

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

- Theorem :

Every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

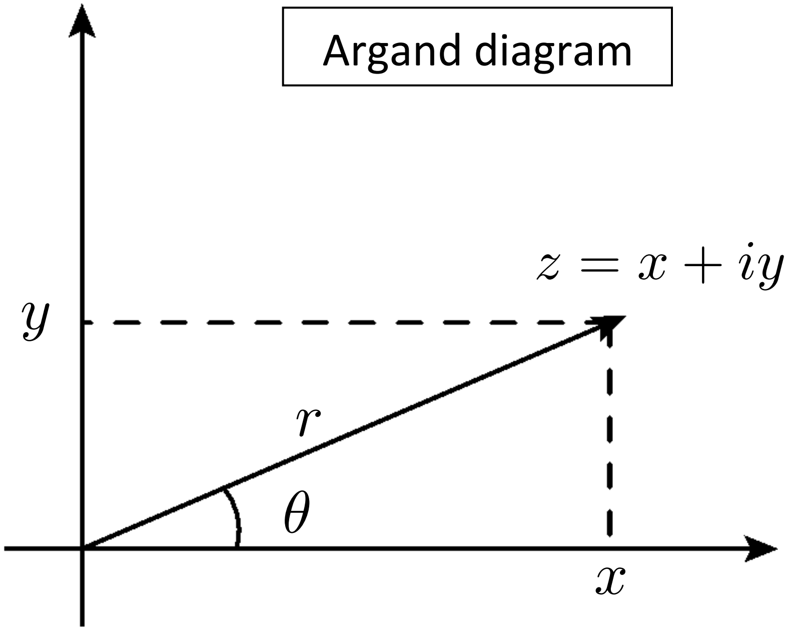
- Consequences :
  - For a given polynomial, one can always find a root in the complex plane. In particular for real polynomials such as x^2+1 (real coefficients, complex roots)
  - Polynomials can be factorized

$$P(z) = \sum_{n=0}^{N} a_n z^z = \prod_{n=0}^{M} (z - z_n)^{r_n}$$

$$\sum_{n=0}^{M} r_n = N$$

This is useful in solving Ordinary Differential Equations, for example

Argand diagram



Square representation :

$$z = x + iy$$

Polar representation :

$$z = re^{i\theta}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$e^{i\pi} + 1 = 0$$

$x = r\cos\theta$ Real part

$y = r\sin\theta$ Imaginary part

$r = |z|$ (modulus, magnitude)

$\theta = \arg(z)$ (argument, phase)

Do it yourself !

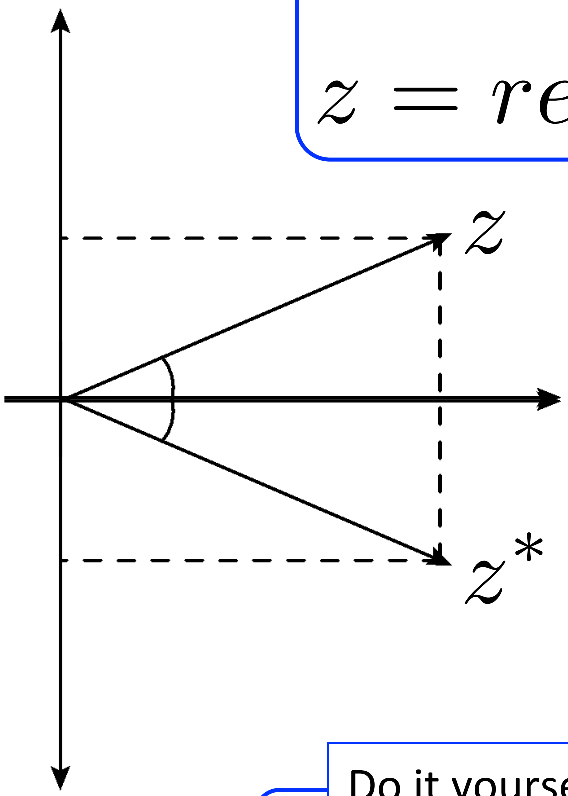
1. Check the equivalence of the square and polar representations, using simple geometrical arguments
2. How would you define the complex exponential function ? (Hint : try to generalize the case of the real exponential)

Complex conjugate

Complex conjugation

$$z = x + iy \qquad z^* = x - iy$$

$$z = re^{i\theta} \qquad z^* = re^{-i\theta}$$



$$|z| = \sqrt{zz^*}$$

$$z \in \mathbb{R} \Leftrightarrow z = z^*$$

- *Real* numbers are fundamental in physics
- Observables are real
- Use of complex numbers can be a convenient mathematical "trick", but in the end, we *observe* or *measure* real quantities
- Example : Hermitian operators

Do it yourself !

1. Check that the product zz* is indeed always positive (so the square root is defined)
2. Prove the equivalence for real numbers

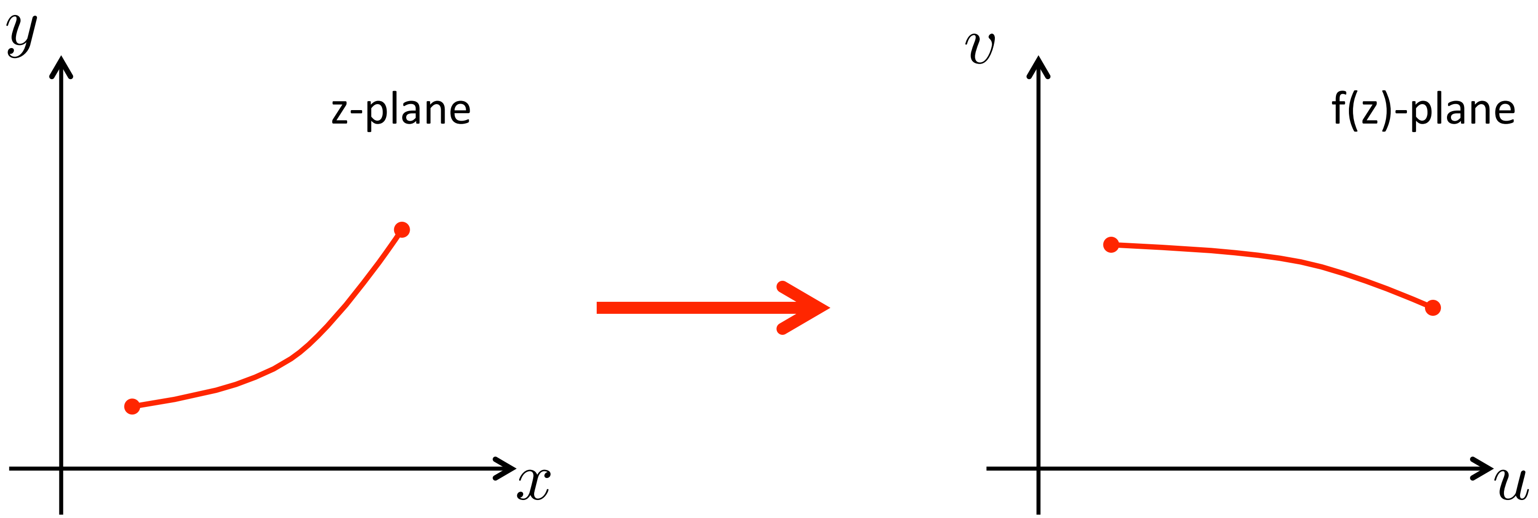
Function of the complex variable

Definition (from real functions)

$$f : \mathcal{A} \subset \mathbb{C} \longmapsto \mathcal{B} \subset \mathbb{C}$$
$$z \longrightarrow f(z)$$

Note : this notation means there is an actual mapping from A to B, which is not always the case (see later)

$$z = x + iy \qquad\qquad f(z) = u(x, y) + iv(x, y)$$



Extension of real (usual) functions

$$f(z) = z^2$$

$$P(z) = \sum_{n=0}^{N} a_n z^n$$

$$f(z) = e^z$$

$$f(z) = \cos z$$

$$f(z) = \ln z$$

$$f(z) = \sin z$$

$$f(z) = z^{\frac{1}{2}}$$

Do it yourself !

1. Which of these functions are single-valued ? Which are not ?
2. Using an analogy with the real case, define sin(z) and cos(z) with the complex exponential
3. Write cos(z) and sin(z) in terms of "usual" cos, sin, cosh, sinh of x and y (real and imaginary parts of z)

Examples of single-valued functions

- Translation $$f(z) = z + z_0$$
- "Rotation" $$f(z) = z z_0$$
- Inversion $$f(z) = \frac{1}{z} \qquad (z \neq 0)$$

Do it yourself !

1. Make a geometrical interpretation of these functions (and justify their names …). Is the rotation a usual rotation ?
2. Check carefully the domain (definition) and codomain of these functions
3. Check that these functions are single-valued in their domain of definition (i.e. to a complex number z corresponds only 1 image by the function f)

Examples of multi-valued functions, cut line

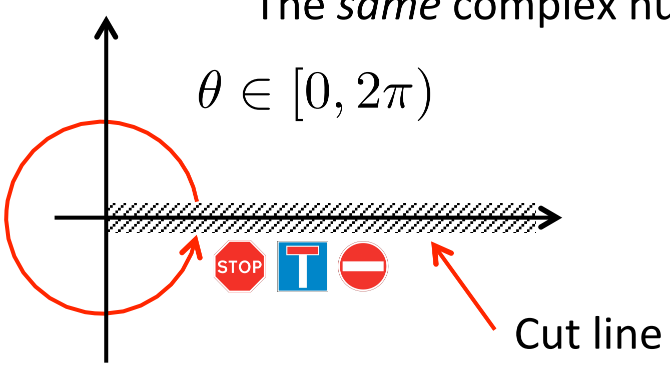
- "Square root" $$f(z) = z^{1/2}$$

- Natural logarithm (ln) $$f(z) = \ln z$$

$$\ln z = \ln(re^{i\theta}) = \ln r + i\theta$$

$$\ln z = \ln(re^{i(\theta+2n\pi)}) = \ln r + i(\theta + 2n\pi)$$

The *same* complex number z leads to many *different* values of f(z) !



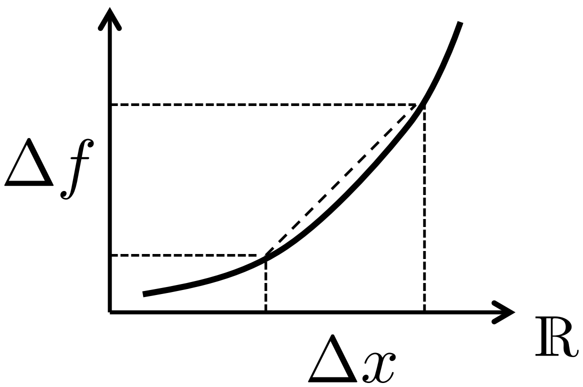
Do it yourself !

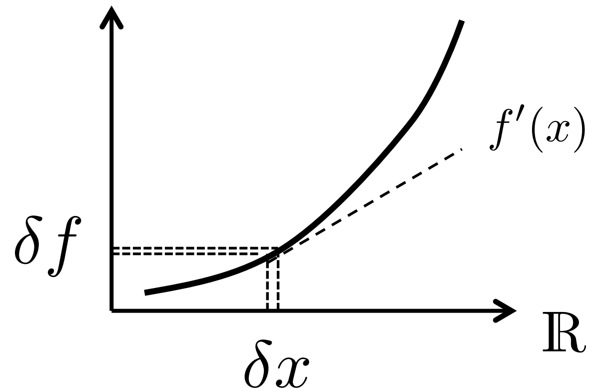
1. Check that the square root function is multi-valued
2. Construct the cut line for the "square root" function
3. Can you find other examples of multi-valued functions ?

- Complex functions can be constructed an extensions of real functions
- Some difficulties may appear
  - Multi-valued functions => cut lines
    - Usually "crossing" a cut line makes the function not continuous (change of phase)
    - Be very careful when studying a function, check for these singularities
  - Singularities (same as the real case)
    - e.g. $f(z) = 1/z$ is not defined in $z = 0$ => singularity
    - We will see more examples later

Derivative of a complex function

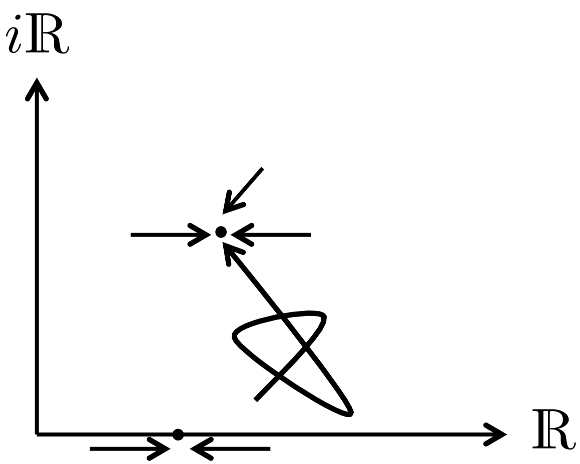
Real function





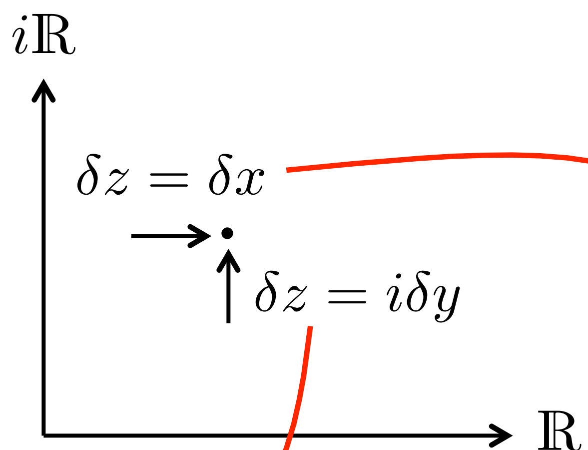
$$f'(x) = \lim_{\delta x \to 0} \frac{\delta f}{\delta x}$$

Complex function



$$f'(z) = \lim_{\delta z \to 0} \frac{\delta f}{\delta z}$$

- provided the limit exists
- and it is independent of the path !



$$f(z) = u(x,y) + iv(x,y)$$

$$\delta f = \delta u + i\delta v$$

$$\delta z = \delta x + i\delta y$$

$$\boxed{\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}}$$

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta y \to 0} \left(\frac{\delta u}{i\delta y} + i \frac{\delta v}{i\delta y} \right) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Cauchy-Riemann conditions (II)

If df/dz exists (is defined), then the Cauchy-Riemann conditions hold :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Conversely : if the Cauchy-Riemann conditions hold, and the partial derivatives of u and v are continuous, then df/dz exists

Cauchy (1789 – 1857)

Riemann (1826 – 1866)

Analytic and entire functions, singularities

- Analytic function
  - *f* is *analytic* at $z = z_0$ if it is differentiable at $z = z_0$ and in some small region around z_0.
  - Synonyms : *holomorphic (*), regular*
  - Important consequences :
    - Analytic functions are infinitely differentiable !
    - Analytic functions can be expressed as a Taylor series

- Entire function
  - *f* is entire if it is *analytic* in the (finite) complex plane

- Singularity
  - z_0 is a singular point if $f'(z)$ is not defined in z_0

(*) Saying that analytic functions are equivalent to holomorphic functions is not simply a matter of semantics. The definition given is actually that of the holomorphic functions. Analytic functions are functions that can be expanded as a power series about a point (Taylor). One can prove that the class of holomorphic functions, and the class of analytic functions, are in fact the same.

Examples

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- Analytic functions :
  - Polynomials
    - (also entire)

$$P_N(z) = \sum_{n=0}^{N} a_n z^n$$

  - Complex exponential
    - (also entire)

$$e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$$

  - Rational functions (ratio of polynomials)
    - (not entire)
    - Need to be “far” from a root of Q(z)

$$R(z) = \frac{P(z)}{Q(z)}$$

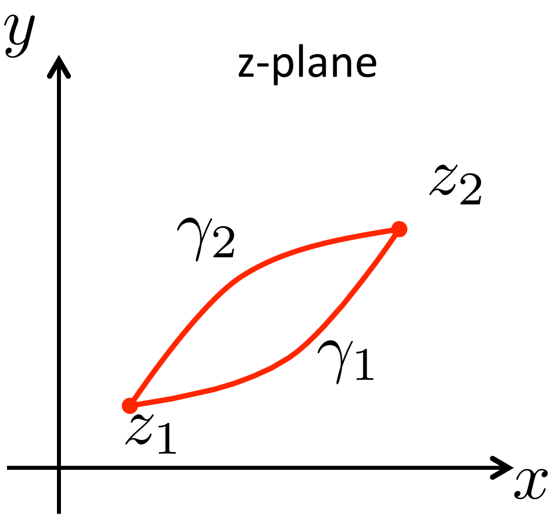
Examples

- Non-analytic functions :
  - Functions involving complex conjugation
    - e.g : $$f(z) = z^*$$
  - Rational functions
    - close to a singularity (root of Q(z))

$$R(z) = \frac{P(z)}{Q(z)}$$

  - Others
    - e.g. $$f(z) = x$$

- Importante note :
  - In the following, we will focus *mainly* on analytic functions
  - Why is that ?
    - Most functions with a physical meaning are analytic, or have a finite number of singularities



$$\gamma = \gamma_1 + \gamma_2$$

- Integration from z_1 to z_2 …
  - Along γ_1 ? $\int_{\gamma_1} f(z)dz$
  - Along γ_2 ? $\int_{\gamma_2} f(z)dz$
  - Same result ?
- Contour ("$\gamma = \gamma_1+\gamma_2$")

$$\oint_{\gamma} f(z)dz$$

Notes :
- The construction of the integral itself is not trivial.
- Like for the real functions, we can construct it step by step and take the limit of a sum – the integral is defined if the sum converges
- The complex integral is closely linked with the 2D integral (plane)

Example

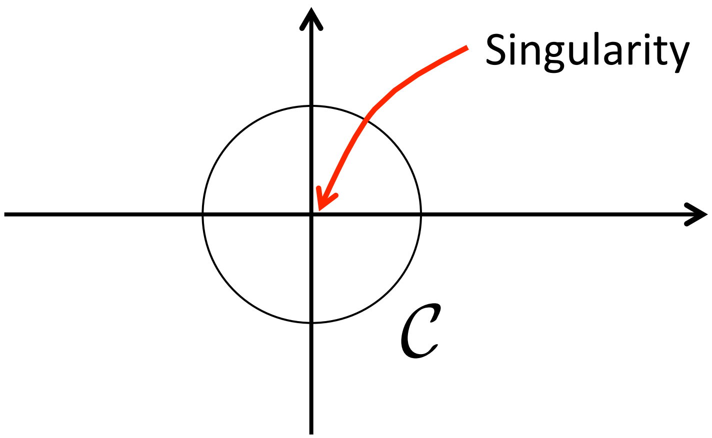
$$\oint_{\mathcal{C}} z^n dz = ?$$

Two cases :

- n ≠ -1

$$\oint_{\mathcal{C}} z^n dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

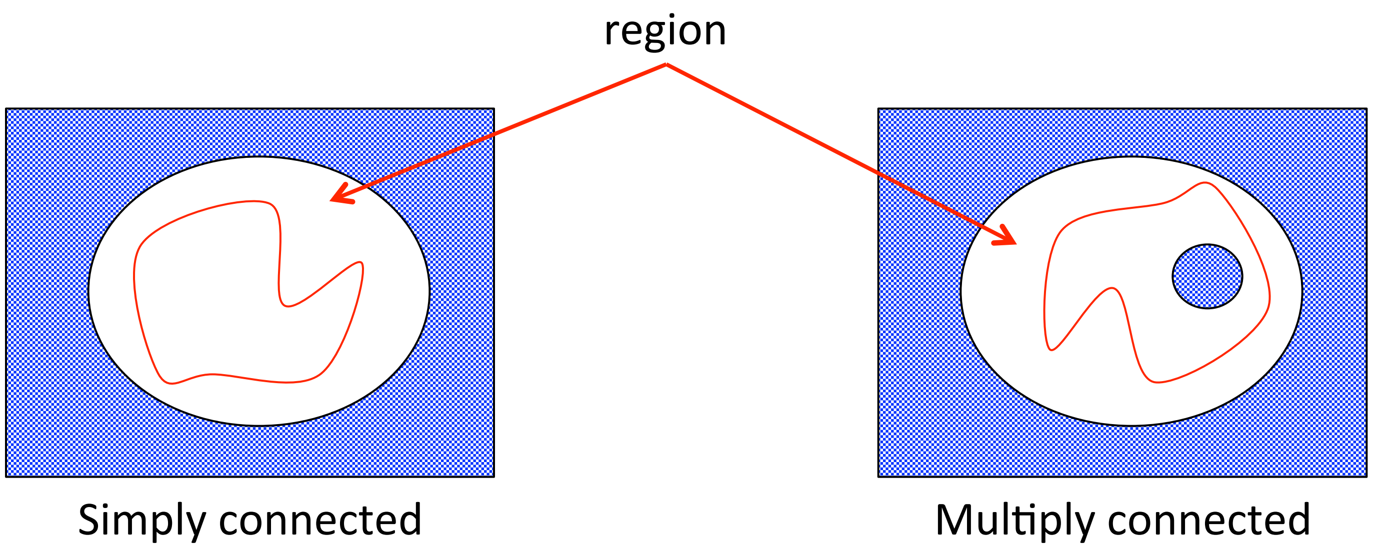
- n = -1



Do it yourself !

1. Why is n=-1 a special case ?
   (Hint : remember the ln(z) function)

Simply connected and multiply connected regions



- Simply connected : every closed contour in the region contains only the points contained in the region (left illustration)
- Multiply connected : a closed contour in the region can contain points that are not contained in the region (right illustration)

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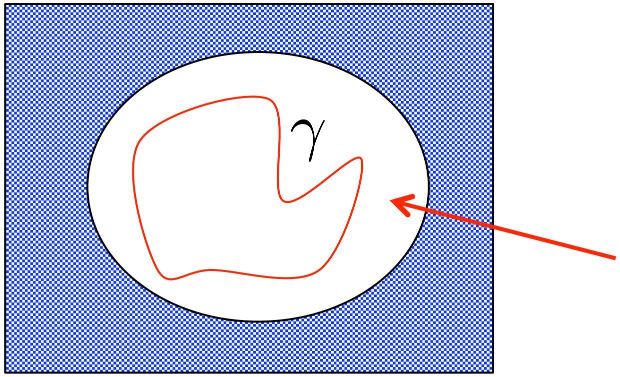
Cauchy Integral Theorem

If f is an analytic function on a simply connected region, and γ a contour contained in it, then :

$$\oint_\gamma f(z)dz = 0$$

Note that the result does not depend on the contour, as long as it is within the simply connected region



f analytic in this region

=> $$\oint_\gamma f(z)dz = 0$$

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Proof of the Cauchy Integral Theorem using Stokes theorem

Step 1 : we decompose *f* and *z* in real and imaginary parts

$$\oint f(z)dz = \oint (u+iv)(dx+idy)$$
$$= \oint \underbrace{(udx - vdy)}_{1} + i \oint \underbrace{(vdx+udy)}_{2}$$

You may recognize a scalar product of a vector function here :

$$\vec{w} \cdot d\vec{l} = w_x dx + w_y dy$$

1 : the case : $\vec{w} = \begin{pmatrix} u \\ -v \end{pmatrix}$

2 : the case : $\vec{w} = \begin{pmatrix} v \\ u \end{pmatrix}$

Therefore 1 and 2 are integrals of the type :

$$\oint \vec{w} \cdot d\vec{l}$$

Stokes theorem !

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Step 2 : Stokes theorem

Infinitesimal surface

Infinitesimal path

$$\int_{\mathcal{S}} (\underbrace{\nabla \times}_{\text{Curl}} \vec{w}) \cdot d\vec{a} = \oint_{\mathcal{P}} \vec{w} \cdot d\vec{l}$$

Surface integral

Vector field (function)

(Closed) path integral

RHS : previous slide !

LHS : let's look at the curl, in 2D …

$$(\nabla \times \vec{w}) \cdot d\vec{a} = \left(\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} \right) dxdy$$

For integral 1 : $\displaystyle \oint (udx - vdy) = \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$

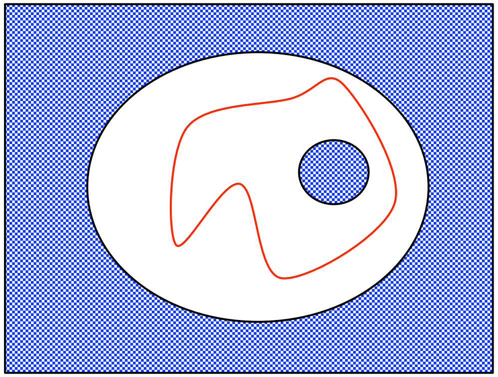
= 0 by Cauchy-Riemann conditions ! (*f* is analytic)

And similarly for integral 2 … also leads to 0 !

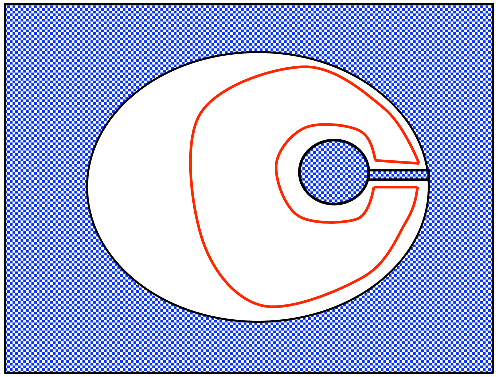
$$\boxed{\oint_{\gamma} f(z)dz = 0}$$

Conversion multiply → simply connected

Is the theorem also valid in a multiply connected region ?



Multiply connected



Simply connected

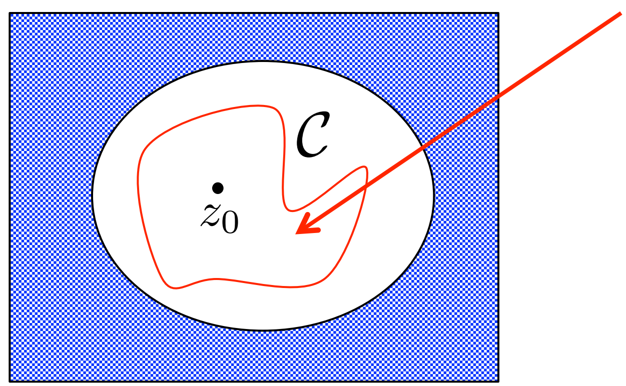
Cauchy's integral formula (I)

Cauchy Integral Formula

If *f* is an analytic function on a simply connected region, and *C* a contour contained in it, then :

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{z-z_0}dz = f(z_0) \qquad (z_0 \text{within the interior of } \mathcal{C})$$

Note that the result does not depend on the contour, as long as it is within the simply connected region



f analytic in this region

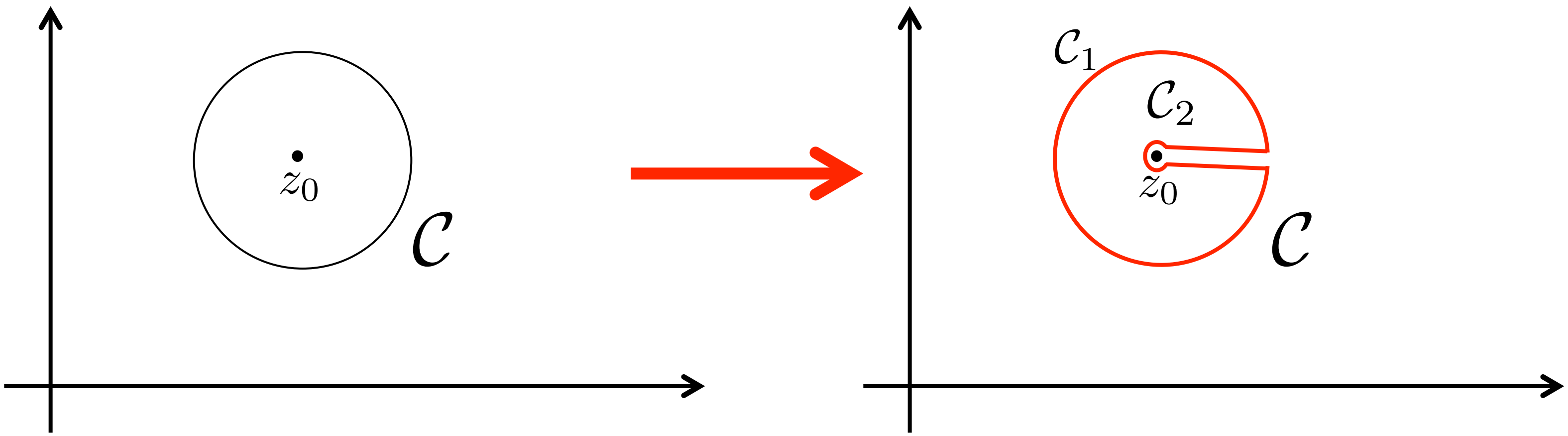
=> $$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{z-z_0}dz = f(z_0)$$

Note : $z \in \mathcal{C} \Rightarrow z - z_0 \neq 0$

The value of an analytic function in a point is determined by the behavior of the function around the point !

Proof using Cauchy's integral theorem

Warning : *f(z)* is analytic, but *f(z)/(z-z₀)* is NOT (in general) ! So, we cannot apply the theorem directly on *C*



We consider C_1 and C_2 oriented counter-clockwise (=> a minus sign !)
The integrals on the straight lines cancel !
Applying Cauchy's integral theorem to "C_1+C_2", we have :

$$\oint_{\mathcal{C}_1} \frac{f(z)}{z-z_0} dz - \oint_{\mathcal{C}_2} \frac{f(z)}{z-z_0} dz = 0$$

Cauchy's integral formula (III)

In the limit where the "cut" region becomes infinitely thin :

- the integral over C_1 becomes the integral over C
- the integral over C_2 becomes :

$$\oint_{\mathcal{C}_2} \frac{f(z)}{z-z_0} dz = \oint_{\mathcal{C}_2} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta$$

On C_2 : $z = z_0 + re^{i\theta}$ $\quad , z_0, r$ constant, $\theta \in [0, 2\pi]$

r is constant, and we take the limit when r goes to 0

$$\oint_{\mathcal{C}_2} \frac{f(z)}{z-z_0} dz = if(z_0) \int_0^{2\pi} d\theta = 2\pi i f(z_0)$$

This leads to :

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z-z_0} dz = f(z_0)$$

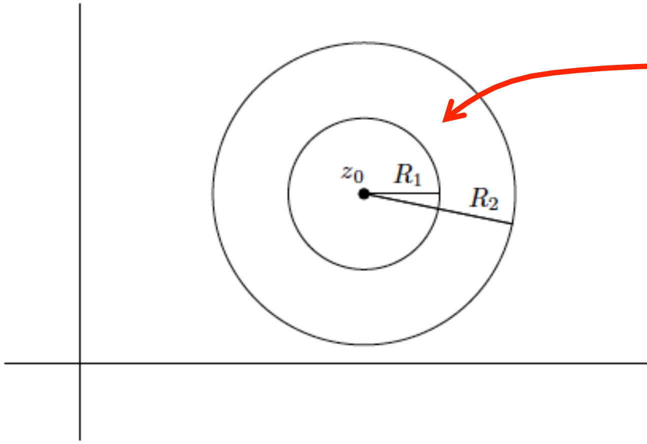
Cauchy's integral formula (IV)

Cauchy's integral formula can be extended to derivatives :

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

We see again that *f* analytic implies that derivatives of all orders exist !

Laurent Series



f analytic inside the annulus

Then we can write *f* as a Laurent series :

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_{\mathcal{C}_r} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Remark:

- In general, *f* may not be defined or differentiable in z_0

$$a_n \stackrel{?}{=} \frac{f^{(n)}(z_0)}{n!}$$

Laurent series are similar to Taylor series, **but may contain negative powers of z !**

Examples

Do it yourself !

1. What is the Laurent expansion of a polynomial ?
2. Give the Laurent expansion of :

$$e^z \qquad e^{\frac{1}{z}} \qquad e^z + e^{\frac{1}{z}}$$

$$\frac{1}{z} + \frac{1}{1-z}$$

Singularities

Definition: a singularity is a point where the function is not differentiable (the function may even not be defined at that point)
A singularity is said **isolated** if the function is differentiable in the neighboring points
Example : f(z) = 1/z

The Laurent expansion of a function around a singularity z_0 is :

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n$$

Definition: if $a_n = 0$ for all $n < -m$, and $a_{-m} \neq 0$, we say that the singularity z_0 is a pole of order m
Examples:

$$f(z) = \frac{1}{z}$$

0 is a pole of order 1

$$f(z) = \frac{1}{(z-1)^3}$$

1 is a pole of order 3

A singularity of "infinite" order is called an **essential singularity**

Residues : the case of 1 singularity

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The Laurent expansion of a function around a singularity z_0 is :

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n$$

Integrating *f* on a circle around the singularity z_0, we have :

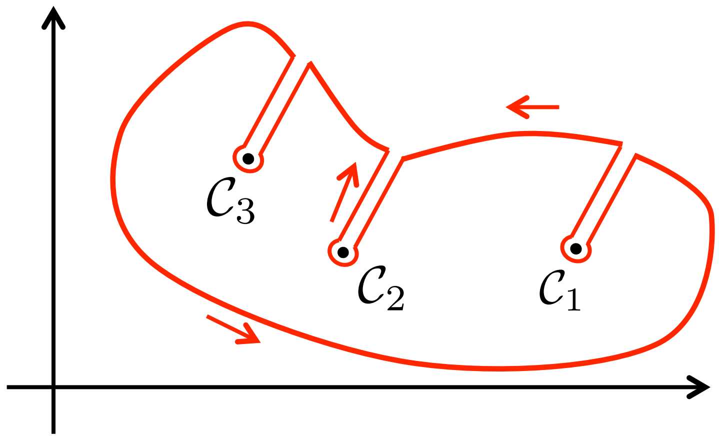
$$a_n \oint_{\mathcal{C}} (z - z_0)^n dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i a_{-1} & \text{if } n = -1 \end{cases}$$

The coefficient a_{-1} is called the residue of *f* at $z = z_0$

$$a_{-1} = \frac{1}{2\pi i} \oint f(z) dz$$

(where the integral path encircles only the singularity z_0)

Residues : the case of several singularities



$$\oint_{\mathcal{C}} f(z)dz + \underbrace{\oint_{\mathcal{C}_1} f(z)dz}_{= -2\pi i a_{-1}|_{z_1}} + \oint_{\mathcal{C}_2} f(z)dz + \oint_{\mathcal{C}_3} f(z)dz + \cdots = 0$$

because counter-clockwise

Cauchy's integral theorem

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The residue theorem

If *f* has a finite number of singularities ("meromorphic"), then :

$$\oint_{\mathcal{C}} f(z)dz = 2\pi i \sum(\text{enclosed residues})$$

Single pole :

$$\mathrm{Res}(f, z_0) = \lim_{z\to z_0} (z - z_0) f(z)$$

Pole of order m :

$$\mathrm{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m f(z))$$

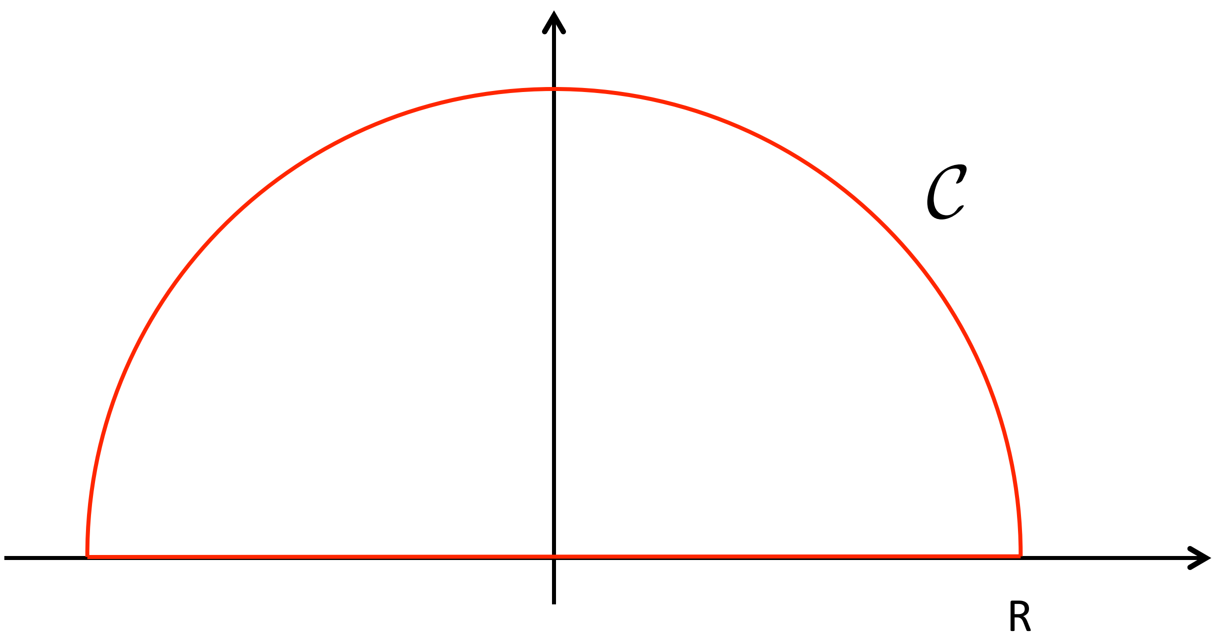
Jordan's Lemma

Jordan (1838 – 1922)

Jordan's lemma

If f is analytic in the upper half-plane except for a finite number of poles, and $\lim_{|z|\to+\infty} f(z) = 0$ $\quad \alpha > 0$

$$\int_{-\infty}^{+\infty} f(x)e^{i\alpha x}dx = \sum \text{Residues (upper half-plane)}$$



This result is very useful for Fourier transforms

Do it yourself !

1. Prove Jordan's lemma
2. What happens if $\alpha < 0$?

Application : the Klein-Gordon operator (I)

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$$D_R(x) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{ip\cdot x}$$

Energy component :

$$\int dE \frac{i}{E^2 - E_p^2} e^{iEt}$$

$E_p^2 = m^2 + \mathbf{p}^2$

Simple poles in $E = \pm E_p$

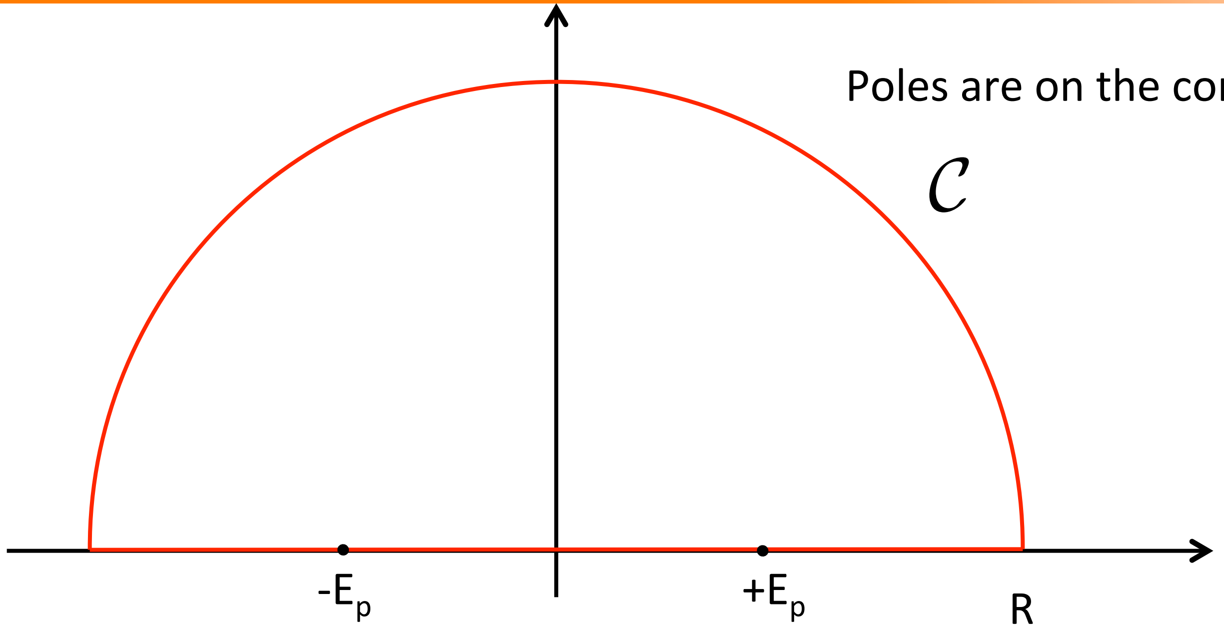
"Feynman trick"

Simple poles in $E = \pm(E_p - \frac{i\epsilon}{2E_p})$

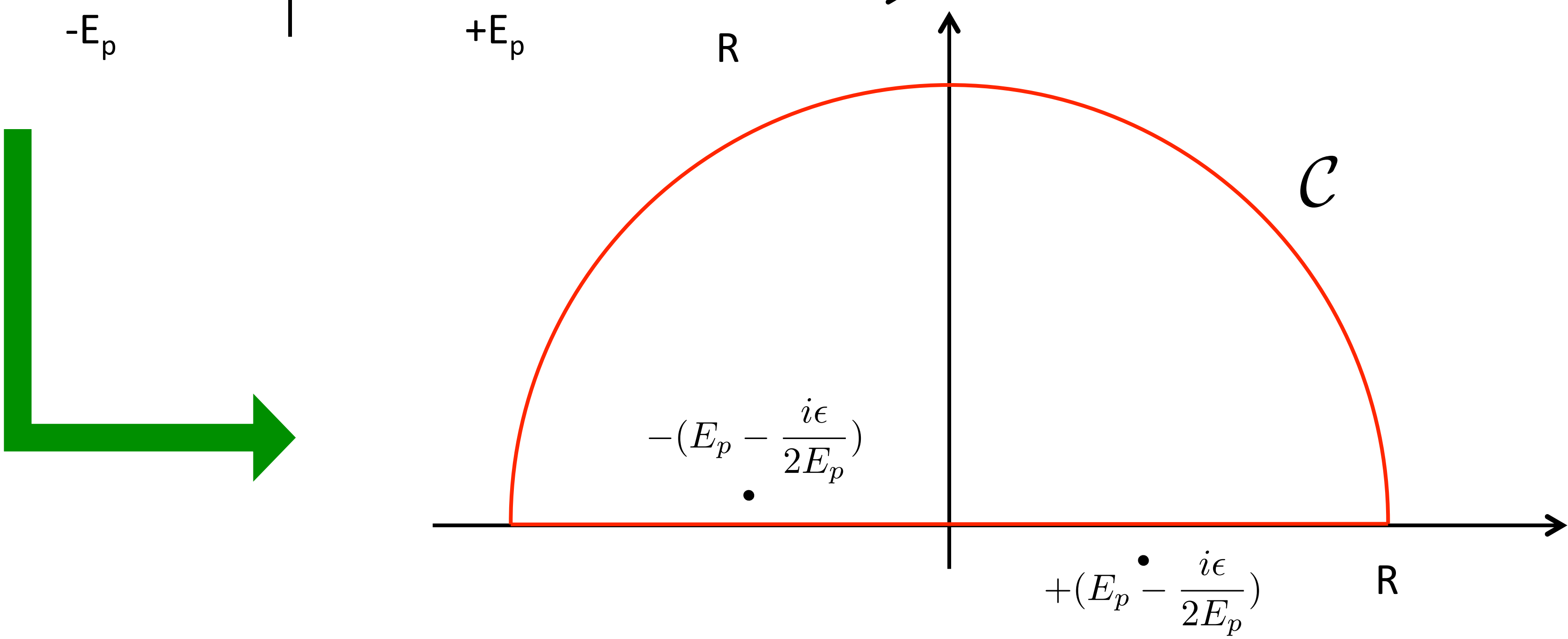
$$\int dE \frac{i}{E^2 - E_p^2 + i\epsilon} e^{iEt} \qquad \epsilon \to 0$$

(neglecting $O(\varepsilon^2)$)

Application : the Klein-Gordon operator (II)



Poles are on the contour !



- What do we have now ?
  - An analytic function with a finite number of poles in the upper half-plane

$$f(E) = \frac{i}{E^2 - E_p^2 + i\epsilon}$$

  - The function vanishes at infinity
  - An integral of the type $\int f(E)e^{iEt}dE$
- => We can apply Jordan's lemma !

Application : the Klein-Gordon operator (IV)

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$$g(E) = f(E)e^{iEt}$$

$$\mathrm{Res}(g, E_0) = \lim_{E \to E_0} (E - E_0)g(E)$$

For : $$E_0 = -(E_p - \frac{i\epsilon}{2E_p})$$

We have : $$\mathrm{Res}(g, E_0) = \frac{e^{-E_p t}}{2E_p}$$ (in the limit $\epsilon \to 0$)

$$D_R(x) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{x}} \frac{e^{-iE_p t}}{E_p}$$